Lagrangian points – Erik Vermaat

This article is a chapter in the EBook "Falling in Space" that is published on the educational resources website http://ngawhetu.nz/ and is published here with permission.

The simplest way to describe the motion of a mass in a gravity field is the so-called two body problem. This can be solved in a closed form (only for point masses) with the Kepler equations. Celestial bodies never are point masses so that gives one complication in the reality of space travel. Another even bigger complication is that there are always other masses in the vicinity that have a non-negligible effect on the motion that is studied.

The strength of gravity diminishes with the square of the distance as Newton taught us, but that means that theoretically the gravity effect of any mass in the Universe only completely vanishes at infinity. In practice we fortunately do not have to incorporate all masses in the Universe when we study e.g. the motion of one of Jupiter's moons, or the orbit of the Earth around the Sun. But we do need to take into account those objects (in our examples within our Solar system) that do have measurable gravitational influence on the motion we study. This leads to at least a three-body problem and in general to an n-body problem in orbital mechanics.

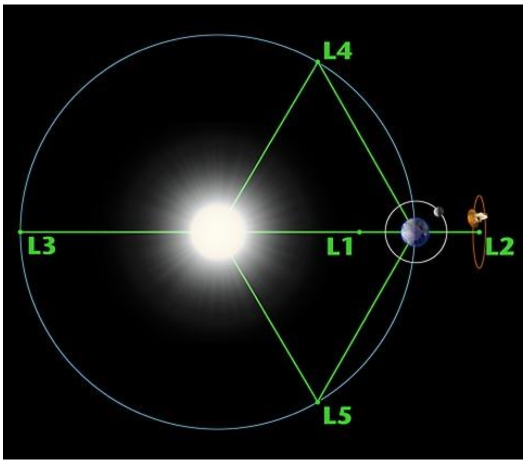
Even the three-body problem cannot be solved in closed form, except in special cases. Nowadays we have fast computers and advanced numerical techniques to solve complex motion problems, and the many successful space missions to date are witness to that. But in the 18th century mathematicians such as Euler and Lagrange were looking for special cases where calculations could be simplified in order to arrive at a solution.

Lagrange studied the three-body problem for the Earth, Sun and Moon and also Jupiter's moons. In 1772 he published a special case solution to this problem and formulated what are now known as Lagrangian points (or Lagrange points). If you add a relatively small mass to a two-body system, there are five locations in that system where the small mass will follow the larger masses in their relative motion.

Definition

The general definition of a Lagrange point is any location in a two-body system where gravity would cause a third (much smaller) object to stay at a constant distance with respect to the two main masses. Lagrange identified five such locations.

Another way Lagrange points are commonly defined is by stating that the combined gravity of the two main bodies produces the exact centripetal force required to follow the orbital motion of the two bodies. However some authors use the term centrifugal force, but it does in our view not seem a good practice to use an imaginary force in a definition. Our first definition closely follows the mathematical formulation and the condition of a constant distance between two points is easily defined in mathematical terms.



L1, L2 and L3 lie on a straight line connection the mass centres of the two masses M1 and M2.
In the Sun-Earth system, L1 and L2 are at a distance of about 1.5 million km from the Earth centre and L3 orbits the Sun at a slightly larger orbit than the Earth, e.g. a little more than 1AU (150 million km) from the Sun.

L4 and L5 are at the same distance from the two masses as the separation between M1 and M2 themselves and thus each form an equilateral triangle with the two masses. This means that L4 and L5 are in the same orbit as the second mass. In case of the Sun-Earth system this is the Earth's orbit, although not precisely because of the slight eccentricity of Earth's orbit.

The importance of these points lies in the fact that any mass (e.g. a spacecraft) located in any of these points will stay in the same position relative to the two masses. So a spacecraft in a Lagrange point of the Sun-Earth system will follow the Earth in its orbit around the Sun.

This is not obvious because Kepler's third law says that an object closer to the central mass (e.g. the Sun) will move faster and objects further out will move slower. In the L1, L2 and L3 points this is certainly not true because objects there will move at the same orbital period as the Earth although they have a different distance to the Sun than Earth. The reason is that the combined gravity of Sun and Earth is "dragging" objects in these locations along with the Earth in its orbit. This is illustrated very well in this ESA website for L1 – L3.

Stability
Objects in L1 and L2 are unstable. This can best be compared with a marble sitting on top of a saddle. At one position the marble is in equilibrium, but the slightest deviation from that point will cause the object to move away. In the case of a mass in any of these Lagrange points the mass will move away exponentially with an e-folding time of about 23 days for the Sun-Earth system. Spacecraft in these locations require periodical position corrections within that frequency. For L3 this time about 150 years.

L4 and L5 are stable when the mass M1 is at least about 25 times that of M2.
For the Sun-Earth system the mass ratio is: 333,000
For the Earth-Moon system it is: 81

Mathematically L4 and L5 are in fact also unstable positions, but because a spacecraft there will essentially orbit the ideal Lagrange point it will thus stay in that region. Compare this with air mass in a hurricane revolving around a vortex which relates to conservation of angular momentum. This will keep the spacecraft close to the Lagrange point for a very long time.

Natural occupants of L-points
Because L4 and L5 are long-term stable (for a large enough mass ratio) asteroids and other natural material can collect at these locations. This is very clear for the Sun-Jupiter system where L4 and L5 are the locations of the Trojan asteroids, the Greek camp in L4 and the Trojan camp in L5. See e.g. this animation.

This could also happen for the Sun-Earth and Earth-Moon systems, although we mostly have only detected some dust concentrations at the Lagrange locations of these systems. In 2010 however the Trojan asteroid 2010 TK7 was detected in L4 of the Sun-Earth system.

Applications

For space missions, some of the Lagrange points are of great significance. Take the Sun-Earth system. The advantage of L1 is an uninterrupted view of the Sun and of L2 it is the fact that the Sun, Earth and the Moon are always “behind” when looking away. The latter is ideal for deep space research and for spacecraft that require to be cooled to low temperature with appropriate light and thermal shielding at the rear of the spacecraft.

The NASA/ESA SOHO spacecraft is in L1 and continuously observes the Sun. This is an early warning mission for Solar activity so continuity is essential. The WMAP and Planck spacecraft were in L2. In 2019 L2 will come again in the spotlight when the James Webb Space Telescope will be positioned there.

An example of use of the L1 point for space travel, is the GRAIL mission. These twin spacecraft were first taken to the Sun-Earth L1 point and from there continued towards the Moon. This was a complicated and time consuming trajectory, but it saved a large amount of fuel, because launching from a Lagrange point requires very little energy and arrival at the moon happened at low speed, saving a lot of fuel for orbit insertion. It took the Apollo astronauts 3 days to get to the Moon while it took the Grail spacecraft 3.5 months.

Some further reading

https://map.gsfc.nasa.gov/mission/observatory_l2.html
http://www.haydenplanetarium.org/tyson/read/2002/04/01/the-five-points-of-lagrange